π -SEMI-PERFECT MODULES

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ABSTRACT

The object of this paper is to study certain class of rings called π −semi-perfect rings and generalizes this concept of modules.

We call a ring R is an π −semi-perfect, if for any element $a \in R$ there is a positive integer n such that $a^n R$ has a complement in R_R, or equivalently, $R/a^n R$ has a projective cover. In the first part, we have got that some of equivalent conditions to concept π − semi –perfect rings.

In the second part, we generalize this concept to projective modules, we have proved that a projective module P is an π −semi-perfect if and only if, endomorphism ring of P is an π −semi-prefect. The main result of this paper the following theorem: a projective module P is an π −semi-perfect if and only if $J(P)$ is small in P and for any $\varphi \in S = End_R(P)$ there is a positive integer n such that $Im\overline{\varphi}^n$ is a direct summand of $\overline{P} = P/J(P)$ and every direct decomposition of $\overline{P}$ can be lifted to a direct decomposition of P .

***Key Words**: Regular ring, π −regular ring, Radical of ring, Complement submodule, Projective module, Projective cover.*

المودولات π- نصف التامة

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الملخص

إن موضوع هذا البحث هو دراسة الحلقات π- نصف التامة وتعميم هذا المفهوم على المـودولات. سوف نقول عن حلقة ما R إنها π - نصف تامة إذا كان لأجل كل عنصر $a \in R$ يوجد عدد صحيح موجب n يحقق $a^n R$ يملك متمماً في R_R والذي بدوره يكافئ أن المودول $R/a^n R$ يملك غطـاء إسقاطياً.

في الجزء الأول من هذا البحث تم التوصل إلى بعض الشروط المكافئة لمفهوم الحلقـة $\pi-$نـصف التامة، في الجزء الثاني منه عمم هذا المفهوم على المودولات الإسقاطية حيث تم إثبات أن الشرط اللازم والكافي كي يكون المودول الإسقاطي P، مودول π-نصف تام هو أن تكون حلقة التشاكلات لـه π- نصف تامة. النتيجة الرئيسية في هذا القسم هي المبرهنة الآتية: الشرط اللازم والكافي كي يكون المودول الإسقاطي P هو مودول $\pi-$نصف تام هو أن يكون الأساس $J(P)$ مودولاً جزئياً صـغيراً فـي P وأنه أياً كان $\varphi \in End_R(P)$ فإنه يوجد عدد صحيح n يحقق $Im\,\bar{\varphi}^n$ هـو حـد مباشـر فـي المودول $\bar{P} = P/J(P)$ وأن كل تفريق مباشر للمودول $\bar{P}$ يمكن أن ينتقل إلـى تفريـق مباشـر للمودول P.

الكلمات المفتاحية: الحلقة المنتظمة، الحلقة π-المنتظمة، أسـاس الحلقـة، المـتمم الجمعي، المودول الإسقاطي، الغطاء الإسقاطي لمودول.

Introduction

Throughout this paper, unless otherwise indicated, all modules over a ring R will be understood to be right $R-$modules. A ring R will always have a unit and every module will be unitary.

Following [1], let R be a ring, M an $R-$module and N a submodule of M. We say that N is small in M if whenever K is a submodule of M with $N+K=M$ then $K=M$. If M is an $R-$module, the radical of M, denoted $J(M)$, is defined to be the intersection of all maximal submodules of M. It may happen that M has no maximal submodules in which case $M=J(M)$ [1].

Thus, for a ring R, $J(R)$ is the Jacobson radical of R. It is easy to show that for any $R-$module M, $J(M)$ coincides with the sum of all small submodules of M [1].

If P is a projective $R-$module then P is a direct summand of a free $R-$ module [2, Theorem 2.2] and hence $J(P)=P.J(R)$. Bass [3, Proposition 2.7] proved that if $P \neq o$ is a projective module then $P \neq P.J(R)$. Thus every projective module has a maximal submodule. A projective cover of $R-$module M is an epimorphism $P \to M$ with small kernel, where P is a projective $R-$ module [4].

Let U and V be a submodules of $R-$module M such that $U+V=M$ and let f be the natural epimorphism $M \to M/U$. Then the restriction of f to V is also epimorphism $V \to M/U$. We call V a complement of U (in M) if the kernel of restriction is minimal i.e., if no proper submodule V' of V satisfies $U+V'=M$. Since the kernel of restriction is $U \cap V$, this equivalent to the condition that $U \cap V$ is small in V [4].

1 - π-Semi-Perfect Rings.

An element a of a ring R is said to be regular (in the sense of Von Neumann) if $a=aba$ for some $b \in R$. If each element of a ring R is regular, R is said to be regular ring [5]. An element a of a ring R is said to be $\pi-$regular if there exists a positive integer n such that $a^n=a^n ba^n$ for some $b \in R$. A ring R is called $\pi-$regular [6], if each element of a ring R is $\pi-$regular.

Now, we call a ring R is an $\pi-$semi-perfect if the factor ring $\overline{R} = R/J(R)$ is an $\pi-$regular and every idempotent of $\overline{R}$ can be lifted to an idempotent of R. The following fact will be needed.

Lemma 1.1. *Let $a \in R$, $a \in J(R)$. Then:*

1 - aR has a complement in R_R .

2 - R/aR has a projective cover.

Proof. 1 – We will prove that R_R is a complement of aR in R_R. It is clear that $R = aR + R$. Let U be a right ideal of R such that $R = aR + U$, since $a \in J(R)$ then $aR \subseteq J(R)$ and $R = J(R) + U$. Since $J(R)$ is small in R it follows that $R = U$. This shows that R_R it is a complement of aR in R_R .

2 – Let $\pi : R_R \to R/aR$ be the natural $R-$epimorphism then $Ker\,\pi = aR \subseteq J(R)$.

Since $J(R)$ is small in R_R follows that $Ker\,\pi$ is small in R_R. This shows that, $R-$epimorphism $\pi : R_R \to R/aR$ is a projective cover of $R-$module R/aR , hence R_R is a projective module.

Proposition 1.2. *For any ring R the following conditions are equivalent:*

1 – R is an $\pi-$semi-perfect ring.

2 – For any $a \in R$ there exists a positive integer n such that $a^n R$ has a complement in R_R which is a direct summand.

3 – For any $a \in R$ there exists a positive integer n such that $R/a^n R$ has a projective cover.

Proof. (1)$\Rightarrow$(2). Let $a \in R$. If $a \in J(R)$ then for any positive integer n , $a^n \in J(R)$ and by lemma 1.1, $a^n R$ has a complement in R_R. Let $a \notin J(R)$ since $\overline{R}$ is an $\pi-$ regular there exists a positive integer n such that the right ideal $\overline{a}^n\overline{R}$ of $\overline{R}$ is generated by an idempotent of $\overline{R}$ which by assumption can be lifted to an idempotent e of R . If we put $e' = 1 - e$, e' is also an idempotent and we have the decomposition $\overline{R} = \overline{e}\overline{R} \oplus \overline{e'}\overline{R}$. Since $a^n\overline{R} = \overline{e}\overline{R}$ follows that $\overline{R} = a^n\overline{R} \oplus \overline{e'}\overline{R}$. Since $e'R$ is a direct summand right ideal, this implies that $e'R$ is a complement of $a^n R$ by [4, lemma 1.5].

(2)$\Rightarrow$(3). Follows immediately from [4, proposition 1.4].

(3)$\Rightarrow$(1). Let $\overline{a} \in \overline{R}$. Then by assumption, there exists a positive integer n such that $R/a^n R$ has a projective cover, by [4, proposition 1.4] $a^n R$ has a complement K in R_R which is a direct

summand of $\bar{R}$ *, by [4, lemma 1.5]* $\bar{R} = \bar{a}^n \bar{R} \oplus \bar{K}$ *. Thus the right ideal* $\bar{a}^n \bar{R}$ *of* $\bar{R}$ *is generated by an idempotent of* $\bar{R}$ *, therefore* $\bar{a}^n \bar{x} \bar{a}^n = \bar{a}^n$ *for some* $\bar{x}$ *of* $\bar{R}$ *, because, since* $\bar{1} \in \bar{R}$ *then there are* $\bar{x} \in \bar{R}, \bar{y} \in \bar{R}$ *such that* $\bar{1} = \bar{a}^n \bar{x} + \bar{y}$ *and* $\bar{a}^n = \bar{a}^n \bar{x} \bar{a}^n + \bar{y} \bar{a}^n$*. We have* $\bar{a}^n, \bar{a}^n \bar{x} \bar{a}^n \in \bar{a}^n \bar{R}$ *, therefore* $\bar{a}^n - \bar{a}^n \bar{x} \bar{a}^n = \bar{y} \bar{a}^n \in \bar{a}^n \bar{R}$ *and* $\bar{y} \bar{a}^n \in \bar{K}$ *, thus* $\bar{a}^n - \bar{a}^n \bar{x} \bar{a}^n = \bar{y} \bar{a}^n \in \bar{a}^n \bar{R} \cap \bar{K} = \{\bar{o}\}$*. Thus,* $\bar{a}^n \bar{x} \bar{a}^n = \bar{a}^n$ *.*

Theorem 1.3. *For any ring* R *the following conditions are equivalent:*

1 - R *is an* $\pi-$*semi-perfect ring.*

2 – For any $a \in R$ *there exists a positive integer* n *and* $e^2 = e \in a^n R$ *such that* $(1-e)a^n \in J(R)$*.*

3 - For any $a \in R$ *there exists a positive integer* n *and* $e^2 = e \in Ra^n$ *such that* $a^n(1-e) \in J(R)$*.*

4 - For any $a \in R$ *there exists a positive integer* n *and* $b \in R$ *such that* $b = ba^n b$ *and* $a^n - a^n ba^n \in J(R)$*.*

Proof. (1)$\Rightarrow$*(2). Let* $a \in R$ *then by proposition 1.2, there exists a positive integer* n *such that* $a^n R$ *has a complement* L *in* R_R *which is a direct summand i.e.,* $R = a^n R + L$ *and* $a^n R \cap L$ *is small in* L *, therefore* $a^n R \cap L \subseteq J(L)$ *Since* L *is a direct summand in* R *then* $J(L) = L \cap J(R)$*. Thus* $a^n R \cap L \subseteq J(R)$

On the other hand, since $R = a^n R + L$ *then by [4, Proposition 1.2] there exists a right ideal* K *of* R *such that* $K \subseteq a^n R$ *and* $R = K \oplus L$ *. Since* K *is a direct summand of* R *then* $K = eR$ *for some idempotent* e *of* R *and* $L = (1-e)R$ *. Thus* $e \in K \subseteq a^n R$ *and* $R = a^n R + (1-e)R$ *. On the other hand,*

$$(1-e)a^n R = a^n R \cap (1-e)R = a^n R \cap L \subseteq J(R)$$

therefore $(1-e)a^n \in J(R)$*.*

(2)$\Rightarrow$*(4). Let* $a \in R$ *then there exists a positive integer* n *and idempotent* $e \in R$ *such that* $e \in a^n R$ *and* $(1-e)a^n \in J(R)$*. Therefore,* $e = a^n r$ *for some* $r \in R$ *. Suppose* $b = ra^n r$ *then* $b = ba^n b$ *and*

$$a^n - a^n ba^n = a^n - a^n ra^n ra^n = a^n - ea^n = (1-e)a^n \in J(R)$$

(4)$\Rightarrow$*(1). Let* $\bar{a} \in \bar{R} = R/J(R)$ *then there exists a positive integer* n *and* $b \in R$ *such that* $a^n - a^n ba^n \in J(R)$ *and* $b = ba^n b$ *. Therefore* $\bar{a}^n \bar{b} \bar{a}^n = \bar{a}^n$ *,* $\bar{R} = R/J(R)$ *is an* $\pi-$*regular ring. Let* $\bar{a}_o$ *is an idempotent in* $\bar{R}$ *then* $b_o = b_o a_o^n b_o$ *for some* $b_o \in R$ *. Suppose* $e = a_o^n b_o$

then e *is an idempotent in* R *and* $\bar{e}=\bar{a}_o^n\bar{b}_o\in\bar{a}_o\bar{R}$. *Thus* $\bar{e}\bar{R}\subseteq\bar{a}_o\bar{R}$. *On the other hand, since* $\bar{a}_o$ *is an idempotent then* $\bar{a}_o=\bar{a}_o^{\,m}$ *for any* $m\in N^*$ *therefore we have* $\bar{a}_o=\bar{a}_o^n=\bar{a}_o^n\bar{b}_o\bar{a}_o^n=\bar{e}\,\bar{a}_o\in\bar{e}\bar{R}$ *and* $\bar{a}_o\bar{R}\subseteq\bar{e}\bar{R}$, *therefore* $\bar{e}\bar{R}=\bar{a}_o\bar{R}$. *Thus,* $\bar{a}_o=\bar{e}\bar{x}=\bar{e}\,\bar{e}\,\bar{x}=\bar{e}\,\bar{a}_o$ *for some* $\bar{x}\in\bar{R}$ *and* $\bar{e}=\bar{a}_o\bar{y}=\bar{a}_o\bar{a}_o\bar{y}=\bar{a}_o\bar{e}$ *for some* $\bar{y}\in\bar{R}$. *We put* $f=e+ea_o(1-e)$ *then* $ef=f$, $fe=e$, $f^2=f$ *and*

$$\bar{f}=\bar{e}+\bar{e}\,\bar{a}_o(\bar{1}-\bar{e})=\bar{e}+\bar{a}_o(\bar{1}-\bar{e})=\bar{e}+\bar{a}_o-\bar{a}_o\bar{e}=\bar{e}+\bar{a}_o-\bar{e}=\bar{a}_o$$

Thus $\bar{R}=R/J(R)$ *is an* $\pi-$*regular ring and every idempotent of* $\bar{R}$ *can be lifted to an idempotent of* R. *By definition* R *is an* $\pi-$*semi-perfect ring.*

(3)$\Rightarrow$*(4). It is proved similarly to (2)*$\Rightarrow$*(4).*

(4)$\Rightarrow$*(3). Let* $a\in R$ *then there exists a positive integer* n *such that* $b=ba^nb$ *and* $a^n-a^nba^n\in J(R)$ *for some* $b\in R$. *Let* $e=ba^n$ *then* e *is an idempotent of* R *and* $e\in Ra^n$. *On the other hand,* $a^n(1-e)=a^n-a^ne=a^n-a^nba^n\in J(R)$. *Thus our proof is completed.*

Lemma 1.4. *If* R *is an* $\pi-$*semi-perfect ring, so is the ring* eRe *for all non-zero idempotent* e *of* R.

Proof. Let R *be an* $\pi-$*semi-perfect ring and let* e *be a non-zero idempotent of* R. *Let* a *be an element of* eRe, *since* $\bar{R}=R/J(R)$ *is* $\pi-$*regular there exists a positive integer* n *and* $r\in R$ *such that* $\bar{a}^n\bar{r}\,\bar{a}^n=\bar{a}^n$. *Since* $a\in eRe$ *then* $a=exe$ *for some* $x\in R$ *and we have*

$$ea^n=e(exe)^n=e\underbrace{(exe)(exe)\cdots(exe)}_{n\ once}=\underbrace{(exe)(exe)\cdots(exe)}_{n\ once}=(exe)^n=a^n$$

similarly, we have $a^ne=a^n$. *Thus,* $ea^n=a^ne=a^n$ *and* $\bar{a}^n\bar{e}\,\bar{r}\,\bar{e}\,\bar{a}^n=\bar{a}^n$.

Since $\bar{e}\,\bar{r}\,\bar{e}\in\bar{e}\bar{R}\bar{e}$ *this shows that the subring* $\bar{e}\bar{R}\bar{e}=eRe/eJ(R)e$ *of* $\bar{R}$ *is an* $\pi-$ *regular ring; here, as is well known,* $eJ(R)e=eRe\cap J(R)$ *is the Jacobson radical of* eRe. *On the other hand, if* $a\in eRe$ *such that* $\bar{a}$ *is an idempotent in* $\bar{R}$ *then by [4, lemma 1.6] there exists an idempotent* $f\in aR$ *such that* $\bar{f}=\bar{e}$. *Since* $ae=ea=a$ *follows that* $fe=f$ *where* $efef=ef^2=ef$ *and* $\overline{ef}=\bar{e}\,\bar{a}=\bar{a}$ *which shows that* $\bar{a}$ *is lifted to the idempotent* $ef\in eRe$.

A ring R *is called* I_0-*ring [7], if any right (left) ideal of* R *is not contained in* $J(R)$, *contains a non-zero idempotent.*

Lemma 1.5. *Any $\pi-$regular ring R with $J(R)=o$ is an I_0-ring.*

Proof. Let R be an $\pi-$regular ring with $J(R)=o$ and let A be a non-zero right ideal of R then there exists $a\in A, a\neq o$. Since R is an $\pi-$regular ring there exists a positive integer n such that $a^n=a^nxa^n$ for some $o\neq x\in R$. Then $e=a^nx$ is an idempotent of R and $e\neq o$, if $e=a^nx=o$ follows $a^n=a^nxa^n=o$, therefore $a\in J(R)=o$, contradict that $a\neq o$. Thus, $e=a^nx\in aR\subseteq A$. This shows, that R is an I_0-ring.

2 - π-Semi-Perfect Modules.

Definition. *We call a module P, is an $\pi-$semi-perfect if P nonzero projective $R-$module and for any $f\in S=End_R(P)$ there exists a positive integer n such that P/Imf^n has a projective cover.*

Proposition 2.1. *For any projective $R-$module P the following conditions are equivalent:*

1 - P is an $\pi-$semi-perfect module.

2 - $S=End_R(P)$ is an $\pi-$semi-perfect ring.

Proof. (1)$\Rightarrow$(2). Let $f\in S$, since P is an $\pi-$semi-perfect there exists a positive integer n such that P/Imf^n has a projective cover by [4, proposition 2.9] S/f^nS has a projective cover by proposition 1.2, we have S which is an $\pi-$ semi-perfect ring.

(2)$\Rightarrow$(1). Let $f\in S$, since S is an $\pi-$semi-perfect ring there exists a positive integer n such that S/f^nS has a projective cover by [2, proposition 2.9] P/Imf^n has a projective cover therefore P is an $\pi-$semi-perfect.

Proposition 2.2. *Every non-zero direct summand of an $\pi-$semi-perfect module is an $\pi-$semi-perfect.*

Proof. Let P be an $\pi-$semi-perfect module and let Q be a non-zero direct summand of P then Q is projective. Let e be the projection of P on to Q then it is easy to see that e is a non-zero idempotent of $S=End_R(P)$, $Q=Ime$ and $End_R(Q)\cong eSe$ (see 4, proposition 2.11). Since P is an $\pi-$semi-perfect module then by proposition 2.1, S is an $\pi-$semi-perfect ring and by lemma 1.4, the ring $End_R(Q)\cong eSe$ is an $\pi-$semi-perfect. Thus, again by proposition 2.1, Q is an $\pi-$semi-perfect module.

Proposition 2.3. *Let P be a projective $R-$module. If P is an $\pi-$semi-perfect then $J(P)$ is small in P.*

Proof. Suppose P, is an $\pi-$semi-perfect, by proposition 2.1, $S=End_R(P)$ is an $\pi-$ semi-perfect ring. Since $S/J(S)$ is an $\pi-$regular ring and $J(S/J(S))=o$ then by lemma 1.5, $S/J(S)$ is an I_0-ring. Since idempotent factor ring $S/J(S)$ can be lifted to an idempotent of S then S is an I_0-ring and by [7, lemma 3.3], $J(P)$ is small in P.

Consider now the factor module $\bar{P}=P/J(P)$ for projective right $R-$ module P. For each submodule U of P we denote by $\bar{U}$ the image of U under the natural epimorphism $P\to\bar{P}$ i.e., $\bar{U}=[U+J(P)]/J(P)$. Since $J(\bar{P})=o$, $\bar{P}$ can be regarded as module over $\bar{R}=R/J(R)$ in the natural manner and $\bar{R}-$ module $\bar{P}$ is projective.

As is well known, there is a one-to-one correspondence between direct decomposition $P=U\oplus V$ and idempotent $e\in S=End_R(P)$ such that e is the projection $P\to U$ (with respect to the decomposition) and conversely U,V are characterized by $U=Ime$, $V=Im(1-e)$. The same, of course, holds between direct decompositions of $\bar{P}$ and idempotent of its endomorphism ring S/H, $H=Hom_R(P,J(P))$ and in this case we have $\bar{U}=Im\,\bar{e}$ and $\bar{V}=Im(\bar{1}-\bar{e})$. This shows that to the decomposition $\bar{P}=\bar{U}\oplus\bar{V}$ there corresponds the idempotent $\bar{e}$. Thus we can conclude that a direct decomposition of $\bar{P}$ can be lifted to a direct decomposition of P, if and only if, the corresponding idempotent of $\bar{S}$ can be lifted to an idempotent of S.

Theorem 2.4. *Let P be a projective $R-$module. Then P is an $\pi-$semi-perfect, if and only if, P satisfies the following three conditions:*

1- $J(P)$ is small in P.

2- For any $\varphi\in S=End_R(P)$ there exists a positive integer n such that $Im\bar{\varphi}^n$ is a direct summand of $\bar{P}$.

3- Every direct decomposition of $\bar{P}$ can be lifted to a direct decomposition of P.

Proof. Assume the condition (1). According to [4, proposition 2.4], this is equivalent to assumption that $J(S)=Hom_R(P,J(P))$, and

$\bar{S} = S/J(S) \cong End_R(\bar{P} = P/J(P))$. *It follows then, from what we have observed above, that the condition (3) is equivalent to the condition that every idempotent of* $S/J(S)$ *can be lifted to an idempotent of* S. *We shall, moreover, show that condition (2) is equivalent to the condition that* $\bar{S} = S/J(S)$ *is an* $\pi-$*regular ring. Suppose that* $\bar{S}$ *is an* $\pi-$*regular ring. Let* $\varphi \in S$ *then there exists a positive integer* n *and* $f \in S$ *such that* $\bar{\varphi}^n \bar{f} \bar{\varphi}^n = \bar{\varphi}^n$ *by [7, lemma 2.1], it follows that* $Im\bar{\varphi}^n$ *is a direct summand of* $\bar{P}$. *Conversely, suppose that* P *satisfies (2). Let* $f \in S$ *then there exists a positive integer* n *such that* $Im\bar{f}^n$ *is a direct summand of* $\bar{P}$. *But, since* $\bar{P}$ *is projective as a right module over* $\bar{R} = R/J(R)$, $Im\bar{f}^n$ *is also projective and therefore the epimorphism* $\bar{f}^n : \bar{P} \to Im\bar{f}^n$ *must split. This means* $Ker\bar{f}^n$ *is a direct summand of* $\bar{P}$, *by [7, lemma 2.1] there exists* $\bar{g} \in \bar{S}$ *such that* $\bar{f}^n \bar{g} \bar{f}^n = \bar{f}^n$. *Thus the endomorphism ring* $\bar{S} = S/J(S)$ *of* $\bar{P}$ *is* $\pi-$ *regular. We have thus seen that the conditions (1),(2) and (3) together imply that* S *is an* $\pi-$*semi-perfect ring and so* P *is an* $\pi-$*semi-perfect module by proposition 2.1.Conversely, suppose* P *is an* $\pi-$*semi-perfect module, then* S *is an* $\pi-$*semi-perfect ring, by proposition 2.1. Therefore we have both the conditions (2),(3), as shown above. Thus our proof is completed.*

Examples.

1 – *Every regular ring is* $\pi-$*regular.*

2–*Let* Q *be an injective* $R-$*module and* $S = End_R(Q)$ *then* $S/J(S)$ *is a regular ring and idempotents factor ring* $S/J(S)$ *can be lifted modulo* $J(S)$, *[8]. Thus endomorphism ring of injective module is* $F-$*semi-perfect.*

3– *A ring* R *is called semi-simple if any right (left) ideal of* R *is a direct summand, [2, Theorem 4.2]. A ring* R *is called artinian if* R *satisfies minimum condition of right (left) ideals of* R. *If* R *is artinian ring then* $R/J(R)$ *is semi-simple. A ring* R *is* $\pi-$*regular, if and only if, every decreasing chain of left (right) ideals of the form* $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots (aR \supseteq a^2R \supseteq a^3R \supseteq \cdots)$ *terminates.*

It is clear that every artinian ring is $\pi-$*regular, but not regular, hence* $J(R) \neq o$.

4– *A commutative* $\pi-$*regular ring with zero Jacobson radical is regular.*

Let R be a commutative $\pi-$regular ring with $J(R)=o$, and $a \in R$. If $a=o$ then $a=axa$ for any $x \in R$, this means that a is a regular element. Suppose that $a \neq o$, since R is $\pi-$regular there exists a positive integer n such that $a^n = a^n ba^n$ for some $b \in R$. On the other hand, $a^n \neq o$ because if $a^n = o$ then $a \in J(R)=o$ contradict that $a \neq o$. Let $e = ba^n$ then $e \neq o$ is an idempotent in R and $1-e \neq 1$ is an idempotent in R. Since $a^n = a^n ba^n = a^n.e$ implies that $a^n(1-e)=o$ and $[a(1-e)]^n = a^n(1-e)^n = a^n(1-e)=o$.

Thus, $a(1-e) \in J(R)=o$ this means that $a=ae$. On the other hand, since $e=ba^n \in Ra$ then $e=ya$ for some $y \in R$, therefore $a=a.e=aya$ this shows that a is a regular element. Thus, R is a regular ring.

5 – *Let R be a $\pi-$regular ring, then the Jacobson radical of R is nil ideal and factor ring $R/J(R)$ is $\pi-$regular. Since $J(R)$ is nil ideal then idempotents factor ring $R/J(R)$ can be lifted modulo $J(R)$, therefore any $\pi-$regular ring is $\pi-$semi-perfect, but not $F-$semi-perfect.*

6 – *R. Ware, [9, Example 3.4] gives example of a regular ring R and a projective regular $R-$module $M=P \oplus Q$ such that $End_R(P) \cong R \cong End_R(Q)$ but $End_R(M)$ is not regular.*

Since $J(R)=o$ it follows that $J(M)=o$ and consequently that $J(End_R(M))=o$. This means that $End_R(M)$ is not $F-$semi-perfect.

7 – *Y. Hirano, [10, Corollary 1] shows that endomorphism ring of finitely generated module over commutative $\pi-$regular ring is $\pi-$regular. This means that if R is a commutative $\pi-$regular ring and M is a finitely generated $R-$module then $End_R(M)$ is $\pi-$semi-perfect, but not $F-$semi-perfect, hence $J(End_R(M)) \neq o$.*

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